MATH 2250 Calculus I
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Practice Test No. 1

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 State the (informal) definition of a limit. (Careful, it's surprising easy to make a logical mistake when rephrasing this definition.)

$\lim_{x \to a} f(x) = L$ if, as x get closer and closer to a, $f(x)$ gets closer and closer to L.

Problem 2 Evaluate the limits below. You do not have to show any work for this problem.

a $\displaystyle\lim_{x \to -2} \frac{x+2}{x^2-4} = \lim_{x \to -2} \frac{x+2}{(x+2)(x-2)} = \lim_{x \to -2} \frac{1}{x-2} = \frac{1}{-4}$

b $\displaystyle\lim_{x \to 4^-} \frac{1}{\sqrt{4-x}} = \infty$

c $\displaystyle\lim_{x \to \infty} \frac{x^2-1}{8x^2-8x} = \frac{1}{8}$

d $\displaystyle\lim_{x \to 3} \frac{x-3}{x^2-x-6} = \lim_{x \to 3} \frac{x-3}{(x-3)(x+2)} = \lim_{x \to 3} \frac{1}{x+2} = \frac{1}{5}$

e $\displaystyle\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

f $\displaystyle\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$

Problem 2 Find the average rate of change of the function $f(x) = 5x^2 + 4$ on the interval $[-1, 2]$.

$$\left(\text{Avg. rate of change}\right) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$= \frac{5(2)^2 + 4 - (5(-1)^2 + 4)}{2 + 1}$$

$$= \frac{24 - 9}{3}$$

$$= 5$$

Problem 3 Prove that the equation $x^3 - e^x = 0$ has a solution on the interval $[0, 3]$. Carefully justify your answer. (Hint: Remember that $2.718 \approx e < 3$).

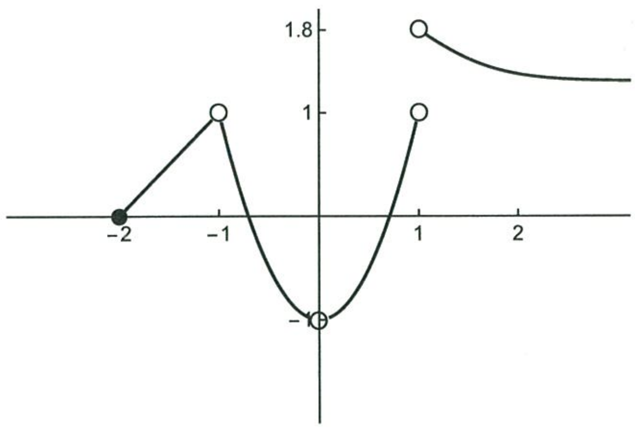
$f(x) = x^3 - e^x$ is continuous, because x^3 is continuous and e^x is continuous. Therefore, we can use the IVT, because $f(0) = 0^3 - e^0 = 0 - 1 = -1$ and $f(3) = 3^3 - e^3 > 0$ because $e < 3$.

Therefore, there is some value c in $[0, 3]$ for which $f(c) = 0$.

Problem 4 State the definition of what it means for a function $f(x)$ to be continuous at a point a (assume that a is not an endpoint of the domain).

$f(x)$ is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$.

Problem 5 Use the below graph to answer this question:



a Find $\lim_{x \to -1} f(x)$. $= 1$

b Find $\lim_{x \to 1^+} f(x)$. $= 1.8$

c Find $\lim_{x \to 1^-} f(x)$. $= 1$

d Find $\lim_{x \to 1} f(x)$. does not exist.

e Is $f(x)$ continuous at $x = -1$? Why or why not? No, because $f(-1)$ does not exist

f Is $f(x)$ continuous at $x = -2$? Why or why not? Yes, because $\lim_{x \to -2^+} f(x) = 0 = f(-$

g Is $f(x)$ continuous? Why or why not? Yes, because $f(x)$ is continuous at every point in its domain.

Problem 4 Evaluate, showing your work:

$$\lim_{x\to 0} 5x^2 + 10x + \sin(x) + e^x$$

The function $f(x) = 5x^2 + 10x + \sin(x) + e^x$ is continuous, so $\lim_{x\to 0} f(x) = f(0)$

$$= 0 + 0 + 0 + 1$$

$$= 1$$

Problem 5 Evaluate, showing your work:

$$\lim_{x\to 1} \frac{-x^2 - 4x}{x^2 + 5x - 4}$$

Rational functions are continuous, and $x = 1$ is in the domain of $f(x) = \frac{-x^2 - 4x}{x^2 + 5x - 4}$, so

$$\lim_{x\to 1} f(x) = f(1) = \frac{-1-4}{1+5-4} = \frac{-5}{2}$$

Problem 6 Evaluate, showing your work:

$$\lim_{x\to 4} \frac{x-4}{x^2 - 3x - 4}$$

Rational functions are continuous, but $x = 4$ is <u>not</u> in the domain of $f(x) = \frac{x-4}{x^2-3x-4}$, so we have to do some work:

$$\lim_{x\to 4} \frac{x-4}{x^2-3x-4} = \lim_{x\to 4} \frac{x-4}{(x-4)(x+1)} = \lim_{x\to 4} \frac{1}{x+1} = \frac{1}{5}$$

Problem 7

a Find the slope of the tangent line (i.e. the instantaneous rate of change) for the function $f(x) = \frac{1}{3x+5}$ at the point $x = 1$, i.e. find

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3(1+h)+5} - \frac{1}{3(1)+5}}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{1}{3h+8} - \frac{1}{8}\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{1}{3h+8}\left(\frac{8}{8}\right) - \frac{1}{8}\left(\frac{3h+8}{3h+8}\right)\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\frac{8-(3h+8)}{8(3h+8)}\right)$$

$$= \lim_{h \to 0} \frac{-3h}{h(8)(3h+8)}$$

$$= \lim_{h \to 0} \frac{-3}{8(3h+8)} = \frac{-3}{8^2} = \frac{-3}{64}$$

b Now find the equation of the tangent line for the function $f(x) = \frac{1}{3x+5}$ at the point $x = 1$

The tangent line to $f(x)$ at $x = 1$ is the line with the slope we found above $\left(\frac{-3}{64}\right)$ that goes through the point $(1, f(1))$, so

$$y - \frac{1}{3(1)+5} = \left(\frac{-3}{64}\right)(x-1)$$ is the equation for the tangent line.

Problem 8

a Find the slope of the tangent line (i.e. the instantaneous rate of change) for the function $f(x) = \sqrt{2x+1}$ at the point $x = 3$, i.e. find

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\sqrt{2(x+h)+1} - \sqrt{2x+1}\right)$$

$$= \lim_{h\to 0} \frac{\sqrt{2(3+h)+1} - \sqrt{2(3)+1}}{h}\left(\frac{\sqrt{2(3+h)+1} + \sqrt{2(3)+1}}{\sqrt{2(3+h)+1} + \sqrt{2(3)+1}}\right)$$

$$= \lim_{h\to 0} \frac{(2(3+h)+1) - (2(3)+1)}{h\left(\sqrt{2(3+h)+1} + \sqrt{2(3)+1}\right)}$$

$$= \lim_{h\to 0} \frac{6+2h+1-6-1}{h\left(\sqrt{2h+7} + \sqrt{7}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2h+7} + \sqrt{7}\right)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2h+7} + \sqrt{7}} = \frac{2}{\sqrt{7}+\sqrt{7}} = \frac{2}{2\sqrt{7}} = \frac{1}{\sqrt{7}}$$

b Now find the equation of the tangent line for the function $f(x) =$ at the point $x = 3$

The tangent line at $x=3$ is given by the line with the slope we found above $\left(\frac{1}{\sqrt{7}}\right)$ that goes through the point $(3, f(3))$, so

$y - \sqrt{2(3)+1} = \left(\frac{1}{\sqrt{7}}\right)(x-3)$ is the equation of the tangent line to $f(x)$ at $x=3$.

Problem 9 Find the value of a such that the function $f(x)$ given below is continuous on the domain $(-\infty, \infty)$. Once you have found the value of a, explain why $f(x)$ is continuous, citing the definition of continuity.

$$f(x) = \begin{cases} a\sin(x), & x \leq \pi/2 \\ ax^2 - 3, & x > \pi/2 \end{cases}$$

For every value of a, $a\sin(x)$ is continuous, and $ax^2 - 3$ is continuous, so we only need to check that $f(x)$ is continuous at $x = \frac{\pi}{2}$.

Set $a\sin\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right)^2 - 3$

$$a = a\left(\frac{\pi}{2}\right)^2 - 3$$

$$a - a\left(\frac{\pi^2}{4}\right) = -3$$

$$a\left(1 - \frac{\pi^2}{4}\right) = -3$$

$$a = \frac{-3}{1 - \frac{\pi^2}{4}}$$

Check: $\lim_{x \to \frac{\pi}{2}} f(x) = a\sin\left(\frac{\pi}{2}\right)$

$= \left(\frac{-3}{1 - \frac{\pi^2}{4}}\right)$

$f\left(\frac{\pi}{2}\right) = a\sin\left(\frac{\pi}{2}\right)$

$= \frac{-3}{1 - \frac{\pi^2}{4}}$

So $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$ ✓

Now check that $\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^-} f(x)$

$\Rightarrow a\sin\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right)^2 - 3$ but this is how we solved for a, so it's true for $a = \frac{-3}{1 - \frac{\pi^2}{4}}$. Now check that $\lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

Problem 10 Find all vertical, horizontal, and slant asymptotes of the following functions:

a $f(x) = \dfrac{3x^2+4x+1}{x^2-1}$

Vertical asymptotes: where $x^2-1=0 \Rightarrow x=1$ and $x=-1$

Horizontal asymptotes: $\lim_{x\to\infty} \dfrac{3x^2+4x+1}{x^2-1} = \dfrac{3}{1} = 3$

and $\lim_{x\to-\infty} \dfrac{3x^2+4x+1}{x^2-1} = \dfrac{3}{1} = 3$,

so $y=3$ is the only horizontal asymptote.

Slant asymptotes: none, because $f(x)$ has a horizontal asymptote.

b $f(x) = \dfrac{x^2-3x+2}{3x-4}$

Vertical asymptotes: where $3x-4=0 \Rightarrow x=\frac{4}{3}$

Horizontal asymptotes: $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} = -\infty$,

so no horizontal asymptotes.

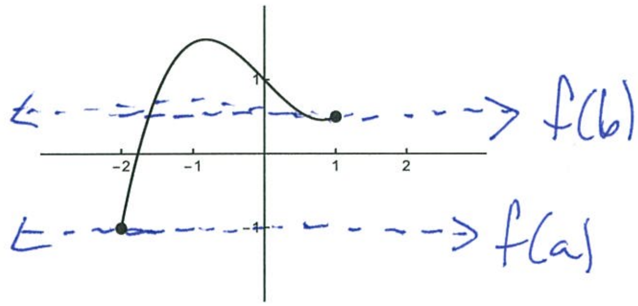
Slant asymptotes:

$$\begin{array}{r l}
 & \frac{1}{3}x - \frac{5}{9} \\
3x-4 \,\big) & x^2-3x+2 \\
 & -\left(x^2-\frac{4}{3}x\right) \\ \hline
 & -\frac{5}{3}x+2 \\
 & -\left(-\frac{5}{3}x+\frac{20}{9}\right) \\ \hline
 & -\frac{2}{9}
\end{array}$$

$\Rightarrow y=\frac{1}{3}x-\frac{5}{9}$ is the only slant asymptote.

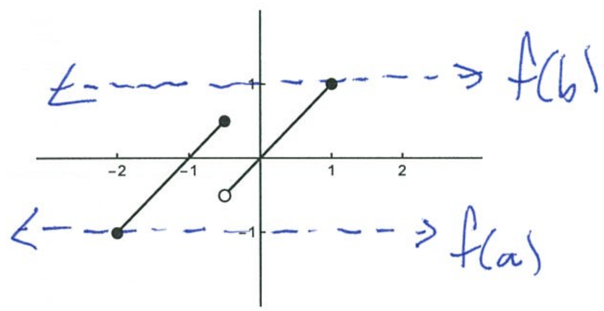
Problem 11 A function $f(x)$ is said to have the *intermediate value property* on an interval $[a, b]$ if, for any value $y = y_0$ between $f(a)$ and $f(b)$, there exists a value $x = c$ such that $f(c) = y_0$. With this definition, we can restate the Intermediate Value Theorem as follows: If $f(x)$ is continuous on $[a, b]$, then $f(x)$ has the intermediate value property on $[a, b]$. Use this to answer the following questions.

a Does the function pictured in the graph below have the intermediate value property on the interval $[-2, 1]$?



Yes. (Check that for any ~~y~~ y_0 between $f(a)$ and $f(b)$ that there is a c such that $f(c) = y_0$)

b Does the function pictured in the graph below have the intermediate value property on the interval $[-2, 1]$?



Yes.

c In general, if a function $f(x)$ has the intermediate value property on the interval $[a, b]$, how does the range of the function $f(x)$ relate to the interval $[f(a), f(b)]$?

The range has to include the interval $[f(a), f(b)]$, but it can include other points as well, e.g. see the graph in part a)